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Exercises for 'Functional Analysis 2' [MATH-404]

(26/05/2025)

Ex 13.1 (Equivalent statements of Brouwer's fixed point theorem*)
Let $\overline{B}_1 := \overline{B_1(0)} \subset \mathbb{R}^n$ be the closed unit ball. Show that the following claims are equivalent :

1) Every continuous map $f : \overline{B}_1 \to \overline{B}_1$ has a fixed point.
2) There exists no continuous map $R : \overline{B}_1 \to \partial B_1$ such that $R(x) = x$ for all $x \in \partial B_1$.
3) Every continuous function $v : \overline{B}_1 \to \mathbb{R}^n$ such that $\langle v(x), x\rangle \leq 0$ for all $x \in \partial B_1$ has a zero in $\overline{B_1(0)}$.

Hint: The implication 2) $\implies$ 1) has been proven in the course. Prove 1) $\implies$ 3) and 3) $\implies$ 2).

Ex 13.2 [Not examinable] (An alternative extension construction)
Let $K \subset \mathbb{R}^n$ be a nonempty, compact and convex set and $f : K \to K$ be continuous. In the lecture we constructed a continuous extension $\widetilde{f} : \mathbb{R}^n \to K$. In this exercise we review a different construction, proposed by a student to Matthias Ruf during the break of the online lectures in 2021.

a) Show that for $x \in \mathbb{R}^n$ there exists a unique point $k(x) \in K$ such that $|x - k(x)| = \inf_{k\in K} |x - k|$.
Hint: For the uniqueness, use that you can equivalently minimize the function $|x - k|^2$ with respect to $k \in K$ and that this function is strictly convex.
b) Show that the map $k : \mathbb{R}^n \to K$, $x \mapsto k(x)$ is continuous.
Hint: Consider $x_j \to x$. Show that any converging subsequence of $k(x_j)$ converges to a minimizer of $k \mapsto |x - k|$, using the minimality of $k(x_j)$.
c) Show that the map $\widetilde{f}(x) = f(k(x))$ for $x \in \mathbb{R}^n$ defines a continuous extension of f to $\mathbb{R}^n$ such that $\widetilde{f}(\mathbb{R}^n) \subset K$. Can you replace compactness of K by a weaker assumption ?

Ex 13.3 (Counterexample to Brouwer's fixed point theorem in infinite dimensions)
Let ℓ^2 be the Banach space of square-summable, real-valued sequences, i.e., $x = (x_i)_{i\in\mathbb{N}} \in \ell^2$ if and only if $x_i \in \mathbb{R}$ and $\|x\|_2^2 := \sum_{i\geq 1} x_i^2 < +\infty$. Set $D = \{x \in \ell^2 : \|x\|_2 \leq 1\}$ and define $f : D \to \ell^2$ by

$$f(x) = (\sqrt{1 - \|x\|_2^2}, x_1, x_2, x_3, \ldots).$$

Show that $f(D) \subset D$, f is continuous, but has no fixed point.

Ex 13.4 (Properties of the subdifferential)
Let $E\colon H \to \mathbb{R}_+ \cup \{\infty\}$ be a convex and lower semicontinuous functional on a real Hilbert space H. We view its (possibly empty, possibly multivalued) subdifferential as a map $\partial^- E\colon H \to 2^H$. Here 2^H denotes the power set of H.

a) Show $\partial^- E$ is a monotone operator. That is, for every $x, y \in H$, every $x^* \in \partial^- E(x)$, and every $y^* \in \partial^- E(y)$,
$$\langle y^* - x^*, y - x\rangle \geq 0.$$

b) Show the following strong-weak closedness property of the graph of $\partial^- E$. Assume $(x_n)_{n\in\mathbb{N}}$ is a sequence in H which converges to $x \in H$. Moreover, let $(x_n^*)_{n\in\mathbb{N}}$ be a sequence of elements $x_n^* \in \partial^- E(x_n)$ weakly converging to $x^* \in H$. Then $x^* \in \partial^- E(x)$.